

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 1972

CRITICAL SHEAR STRESS OF A-CURVED RECTANGULAR
PANEL WITH A CENTRAL STIFFENER

By Manuel Stein and David J. Yaeger

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SUMMARY

A theoretical solution is given for the critical shear stress of a simply supported curved rectangular plate stiffened by a central stiffener offering no torsional restraint which lies in either the axial or circumferential direction. Results are obtained by means of the Galerkin method and are presented in the form of computed curves and tables.

INTRODUCTION

As part of an investigation to determine whether the critical load of a curved rectangular panel can be significantly increased by means of a centrally located stiffener, a panel under axial-compressive load having a central circumferential stiffener of zero torsional stiffness was treated in reference 1; a panel having a central axial stiffener under the same loading conditions was treated in reference 2. These two papers enable a designer to determine the more effective way of reinforcing curved rectangular panels with a single central stiffener to resist axial compression. The purpose of the present paper is to enable the designer to determine the most effective way of stiffening the same type of panel in shear.

The stiffeners are assumed to have bending stiffness but no torsional stiffness and are assumed to be concentrated along axial or circumferential lines in the middle plane of the panel. Because the critical shear stress is, in general, nearly independent of boundary conditions for curved panels, except when the panels are very long in the axial direction, only the simply supported case has been investigated.

The results of the analysis are presented in the form of curves and tables from which the critical stress for a stiffened panel may be

determined if the dimensions are known. The analysis, which is based on the Galerkin method, is given in appendixes A and B. The theoretical solution for a curved rectangular panel with a central circumferential stiffener is given in appendix A, and the theoretical solution for a curved rectangular panel with a central axial stiffener is given in appendix B.

The panel proportions considered cover the range from nearly flat plates to highly curved plates, with aspect ratios of 1, 1.5, and 2.

SYMBOLS

a	longer dimension of panel
b	shorter dimension of panel
m, n, p, q	integers
r	radius of curvature of panel
t	thickness of panel
w	displacement in radial direction of point in median surface of panel; positive outward
x	axial coordinate of panel
y	circumferential coordinate of panel
D	flexural stiffness of panel per unit length $\left(\frac{Et^3}{12(1 - \mu^2)} \right)$
E	Young's modulus of elasticity
I	effective moment of inertia of stiffener
Z	curvature parameter $\left(\frac{b^2}{rt} \sqrt{1 - \mu^2} \right)$
a_{kq}, a_{mn}, a_{pq}	deflection coefficients in trigonometric series
k_s	critical-shear-stress coefficient appearing in formula $\tau = \frac{k_s \pi^2 D}{b^2 t}$

$$M_{pq} = \frac{\pi^2}{32\beta^3 k_s} \left[(p^2 + q^2\beta^2)^2 + \frac{12\beta^4 p^4 z^2}{\pi^4 (p^2 + q^2\beta^2)^2} \right]$$

β aspect ratio $\left(\frac{a}{b}\right)$

μ Poisson's ratio

τ critical shear stress

Q operator defined in appendixes

$\delta\left(x - \frac{a}{2}\right)$ Dirac delta function defined by

$$\int_{x_1}^{x_2} f(x) \delta\left(x - \frac{a}{2}\right) dx = f\left(\frac{a}{2}\right) \text{ where } x_1 < \frac{a}{2} < x_2$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

∇^{-4} inverse of ∇^4 defined by $\nabla^{-4}(\nabla^4 w) = w$

RESULTS AND DISCUSSION

The critical shear stress of a simply supported curved rectangular panel with a centrally located axial or circumferential stiffener having zero torsional stiffness (see fig. 1) is given by the formula

$$\tau = k_s \frac{\pi^2 D}{b^2 t}$$

where k_s is a numerical coefficient plotted in figure 2 as a function of the aspect ratio of the panel, the panel curvature, and the flexural stiffness of the stiffener.

The horizontal cut-offs in the curves of figure 2 represent approximately the maximum possible buckling strength obtainable through

an increase in stiffener flexural stiffness. They were calculated by assuming that when the stiffeners are flexurally rigid, the section of plate between stiffeners buckles as a simply supported plate. These cut-offs are somewhat conservative because of the strengthening effect of continuity of the sheet across the stiffeners. (See reference 3.)

The maximum possible increase in shear buckling strength obtainable through the use of a centrally located stiffener is shown in table 1. In this table the buckling stress coefficients k_s are given for plates with no stiffener and for the same plates with an infinitely stiff stiffener. (The plates with infinitely stiff stiffeners will necessarily have nodes at the stiffeners and are represented by the cut-offs in fig. 2.) The percentage increase in k_s due to the stiffener is noted.

Both table 1 and figure 2 show that with either a central axial or central circumferential stiffener the percentage increase in strength of the stiffened panel over the unstiffened panel decreases as the curvature parameter Z increases. The reduction of stiffener effectiveness as the curvature parameter increases is not so marked in the present case as it is for the case of axial compression (references 1 and 2).

For a panel with a stiffener in the longer direction the percentage increase in strength increases with aspect ratio, but for a panel with a stiffener in the shorter direction the percentage increase in strength decreases with aspect ratio.

CONCLUDING REMARKS

A theoretical solution is given for the shear buckling stress of a curved rectangular panel stiffened by a central axial or circumferential stiffener. The results are presented in curves and tables and show that the effectiveness of a stiffener in raising the shear buckling stress is reduced as the curvature parameter increases. The reduction in effectiveness for shear, however, is not so marked as it is in the case of axial compression.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., August 26, 1949

APPENDIX A

THEORETICAL SOLUTION FOR CRITICAL SHEAR STRESS
OF CURVED RECTANGULAR PANEL WITH CENTRAL
CIRCUMFERENTIAL STIFFENER

Equation of equilibrium.— The critical shear stress of a curved rectangular panel having a central circumferential stiffener of zero torsional stiffness located at $x = \frac{a}{2}$ may be obtained by solving the equation of equilibrium

$$D\nabla^4 w + \frac{Et}{r^2} \nabla^4 \frac{\partial^4 w}{\partial x^4} + EI \frac{\partial^4 w}{\partial y^4} \delta\left(x - \frac{a}{2}\right) + 2\tau t \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (A1)$$

This equation, without the third term, is discussed in reference 4. The third term which represents the effect of the stiffener flexural stiffness is explained in reference 1.

The equation of equilibrium may be represented by

$$Q(w) = 0 \quad (A2)$$

Method of solution.— Equation (A1) may be solved by the Galerkin method as outlined in references 4 and 5. As suggested in reference 4 for simply supported curved rectangular panels, the following series expansion is used for w :

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (A3)$$

The coordinate system used is shown in figure 1(a). The coefficients a_{mn} are then chosen to satisfy the equation

$$\int_0^a \int_0^b \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} Q(w) dx dy = 0 \quad \begin{matrix} (p = 1, 2, 3 \dots) \\ (q = 1, 2, 3 \dots) \end{matrix} \quad (A4)$$

When the operations indicated in equation (A4) are performed, a set of homogeneous linear algebraic equations in the unknown coefficients a_{pq} is obtained with k_g appearing as a parameter. The solution for the critical-shear-stress coefficient k_g is then found to be the minimum value of k_g for which the algebraic equations have a nonvanishing solution for a_{pq} , that is, for which the plate is in equilibrium in a deflected state.

Substitution of the expressions for Q and w given by equations (A2) and (A3) into equation (A4) leads to the following set of algebraic equations:

$$\begin{aligned} a_{pq} & \left[(p^2 + q^2 \beta^2)^2 + \frac{12Z^2 p^4 \beta^4}{\pi^4 (p^2 + q^2 \beta^2)^2} \right] \\ & + \frac{32k_g}{\pi^2} \beta^3 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} \\ & + 2 \frac{EI}{Db} q^4 \beta^3 \sin \frac{p\pi}{2} \sum_k a_{kq} \sin \frac{k\pi}{2} = 0 \quad \begin{matrix} (p = 1, 2, 3 \dots) \\ (q = 1, 2, 3 \dots) \end{matrix} \end{aligned} \quad (A5)$$

where the summation includes only those values of m and n for which $m \pm p$ and $n \pm q$ are odd. The condition for a nonvanishing solution of these equations is the vanishing of the determinant of the coefficients of the unknowns a_{mn} . This infinite determinant can be factored into the product of two infinite subdeterminants, one in which $p \pm q$ is even and one in which $p \pm q$ is odd. The vanishing of these subdeterminants leads to the following determinantal equations:

If $p \pm q$ is even,

	a_{11}	a_{13}	a_{22}	a_{31}	a_{33}	a_{24}	a_{42}	a_{15}	a_{51}	a_{44}	...
$p = 1, q = 1$	M_{11}	0	$\frac{4}{9}$	$\frac{\pi^2 EI}{16k_s Db}$	0	$\frac{8}{45}$	$\frac{8}{45}$	0	$\frac{\pi^2 EI}{16k_s Db}$	$\frac{16}{225}$...
$p = 1, q = 3$	0	M_{13}	$-\frac{4}{5}$	0	$\frac{-81\pi^2 EI}{16k_s Db}$	$\frac{8}{7}$	$-\frac{8}{25}$	0	0	$\frac{16}{35}$...
$p = 2, q = 2$	$\frac{4}{9}$	$-\frac{4}{5}$	M_{22}	$-\frac{4}{5}$	$\frac{36}{25}$	0	0	$-\frac{20}{63}$	$-\frac{20}{63}$	0	...
$p = 3, q = 1$	$\frac{\pi^2 EI}{16k_s Db}$	0	$-\frac{4}{5}$	M_{31}	0	$-\frac{8}{25}$	$\frac{8}{7}$	0	$\frac{\pi^2 EI}{16k_s Db}$	$\frac{16}{35}$...
$p = 3, q = 3$	0	$\frac{-81\pi^2 EI}{16k_s Db}$	$\frac{36}{25}$	0	M_{33}	$-\frac{72}{35}$	$-\frac{72}{35}$	0	0	$\frac{144}{49}$...
$p = 2, q = 4$	$\frac{8}{45}$	$\frac{8}{7}$	0	$-\frac{8}{25}$	$-\frac{72}{35}$	M_{24}	0	$-\frac{40}{27}$	$-\frac{8}{63}$	0	... = 0
$p = 4, q = 2$	$\frac{8}{45}$	$-\frac{8}{25}$	0	$\frac{8}{7}$	$-\frac{72}{35}$	0	M_{42}	$-\frac{8}{63}$	$-\frac{40}{27}$	0	... (A6)
$p = 1, q = 5$	0	0	$-\frac{20}{63}$	0	0	$-\frac{40}{27}$	$-\frac{8}{63}$	M_{15}	0	$-\frac{16}{27}$...
$p = 5, q = 1$	$\frac{\pi^2 EI}{16k_s Db}$	0	$-\frac{20}{63}$	$\frac{\pi^2 EI}{16k_s Db}$	0	$-\frac{8}{63}$	$-\frac{40}{27}$	0	M_{51}	$-\frac{16}{27}$...
$p = 4, q = 4$	$\frac{16}{225}$	$\frac{16}{35}$	0	$\frac{16}{35}$	$\frac{144}{49}$	0	0	$-\frac{16}{27}$	$-\frac{16}{27}$	M_{44}	...
.
.
.

and if $p \pm q$ is odd,

	a_{12}	a_{21}	a_{14}	a_{23}	a_{32}	a_{41}	a_{16}	a_{25}	a_{34}	a_{43}	...
$p = 1, q = 2$	M_{12}	$-\frac{1}{9}$	0	$\frac{1}{5}$	$-\frac{2}{15}$	$-\frac{8}{45}$	0	$\frac{20}{63}$	0	$\frac{8}{25}$...
$p = 2, q = 1$	$-\frac{1}{9}$	M_{21}	$-\frac{8}{45}$	0	$\frac{1}{5}$	0	$-\frac{1}{35}$	0	$\frac{8}{25}$	0	...
$p = 1, q = 4$	0	$-\frac{8}{45}$	M_{14}	$-\frac{8}{7}$	0	$-\frac{16}{225}$	0	$\frac{40}{27}$	$-\frac{16}{45}$	$-\frac{16}{35}$...
$p = 2, q = 3$	$\frac{1}{5}$	0	$-\frac{8}{7}$	M_{23}	$-\frac{36}{25}$	0	$-\frac{1}{9}$	0	$\frac{72}{35}$	0	...
$p = 3, q = 2$	$-\frac{2}{15}$	$\frac{1}{5}$	0	$-\frac{36}{25}$	M_{32}	$-\frac{8}{7}$	0	$-\frac{1}{7}$	0	$\frac{72}{35}$...
$p = 4, q = 1$	$-\frac{8}{45}$	0	$-\frac{16}{225}$	0	$-\frac{8}{7}$	M_{41}	$-\frac{8}{175}$	0	$-\frac{16}{35}$	0	...
$p = 1, q = 6$	0	$-\frac{1}{35}$	0	$-\frac{1}{9}$	0	$\frac{8}{175}$	M_{16}	$-\frac{20}{11}$	0	$-\frac{8}{45}$...
$p = 2, q = 5$	$\frac{20}{63}$	0	$\frac{40}{27}$	0	$-\frac{1}{7}$	0	$-\frac{20}{11}$	M_{25}	$-\frac{8}{3}$	0	...
$p = 3, q = 4$	0	$\frac{8}{25}$	$-\frac{16}{45}$	$\frac{72}{35}$	0	$-\frac{16}{35}$	0	$-\frac{8}{3}$	M_{34}	$-\frac{135}{49}$...
$p = 4, q = 3$	$\frac{8}{25}$	0	$-\frac{16}{35}$	0	$\frac{72}{35}$	0	$-\frac{8}{45}$	0	$-\frac{135}{49}$	M_{43}	...
.
.
.

= 0 (A7)

where

$$M_{pq} = \frac{p^2}{32p^3q} \left[(p^2 + q^2p^2)^2 + \frac{12p^2p^4q^4}{p^2 + q^2p^2} \right]$$

if p is even and

$$M_{pq} = \frac{p^2}{32p^3q} \left[(p^2 + q^2p^2)^2 + \frac{12p^2p^4q^4}{p^2 + q^2p^2} + 2q^4p^3 \frac{p}{q} \right]$$

if p is odd.

These determinants give the buckling load of a curved rectangular panel stiffened by a central circumferential stiffener with various length-width ratios for buckle patterns respectively symmetrical and antisymmetrical about the center of the panel.

By use of a finite determinant including the rows and columns corresponding to the most important terms in the expansion for w (equation (A3)), equations (A6) and (A7) were solved by the Crout method (reference 6) for the lowest value of k_s which satisfied these equations. The lower of the two values of k_s found by solving equations (A6) and (A7) is the critical-shear-stress coefficient for the particular values of β and Z under consideration.

The results are plotted in the form of the critical-shear-stress coefficient against the ratio of the flexural stiffness of the stiffener to that of the plate for different values of β and Z . For purposes of comparison, the symbols defining the circumferential and axial dimensions are interchanged in table 1 and in figure 2.

APPENDIX B

THEORETICAL SOLUTION FOR CRITICAL SHEAR STRESS
OF CURVED RECTANGULAR PANEL WITH
CENTRAL AXIAL STIFFENER

Equation of equilibrium.— The critical shear stress of a curved rectangular panel reinforced by a central axial stiffener of zero torsional stiffness located at $y = \frac{b}{2}$ may be obtained by a method of solution almost identical with that employed in appendix A.

The new equation of equilibrium becomes

$$D\nabla^4 w + \frac{Et}{r^2} \nabla^4 \frac{\partial^4 w}{\partial x^4} + EI \frac{\partial^4 w}{\partial x^4} \delta\left(y - \frac{b}{2}\right) + 2\tau t \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (B1)$$

This equation, in turn, may be represented by

$$Q(w) = 0 \quad (B2)$$

Method of solution.— Equation (B2) may be solved by the Galerkin method in a manner similar to that used in appendix A, where the following series expansion is used for w :

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (B3)$$

The coordinate system used is shown in figure 1(b). The coefficients a_{mn} are then chosen to satisfy the equation

$$\int_0^a \int_0^b \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} Q(w) dx dy \quad \begin{matrix} (p = 1, 2, 3 \dots) \\ (q = 1, 2, 3 \dots) \end{matrix} \quad (B4)$$

Substitution of the expressions for Q and w given by equations (B2) and (B3) into equation (B4) leads to the following set of algebraic equations:

$$\begin{aligned}
 a_{pq} \left[(p^2 + q^2 \beta^2)^2 + \frac{12Z^2 \beta^4 p^4}{\pi^4 (p^2 + q^2 \beta^2)^2} \right] \\
 + \frac{32k_s \beta^3}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} \\
 + 2 \frac{EI}{D_b} p^4 \sin \frac{q\pi}{2} \sum_k a_{pk} \sin \frac{k\pi}{2} = 0 \quad \begin{matrix} (p = 1, 2, 3 \dots) \\ (q = 1, 2, 3 \dots) \end{matrix} \quad (B5)
 \end{aligned}$$

where the summation includes only those values of m and n for which $m \pm p$ and $n \pm q$ are odd. The condition for a nonvanishing solution of these equations is the vanishing of the determinant of the coefficients of the unknowns a_{mn} . This infinite determinant can be factored into the product of two infinite subdeterminants similar to those in appendix A, one in which $p \pm q$ is even and one in which $p \pm q$ is odd.

These determinants give the buckling load of a rectangular curved panel stiffened by a central axial stiffener with various length-width ratios for buckle patterns respectively symmetrical and antisymmetrical about the center of the panel.

Using a finite determinant, such as that employed in appendix A, and solving by the Crout method for the lowest value of k_s which satisfies these equations result in the critical-shear-stress coefficients for the particular values of β and Z under consideration.

The results are plotted in the same way as are those found in appendix A. Figure 2 shows the comparison between axial and circumferential stiffening of curved rectangular panels.

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TABLE 1

CRITICAL-SHEAR-STRESS COEFFICIENTS FOR BUCKLING WITHOUT STIFFENER
AND FOR BUCKLING WITH NODE AT STIFFENER OFFERING
ZERO TORSIONAL RESTRAINT

b/a	$z = \frac{b^2}{rt} \sqrt{1 - \mu^2}$	$k_s = \frac{rtb^2}{D\pi^2}$		Percentage increase $\frac{(b) - (a)}{(a)} \times 100$
		(a) Buckling without stiffener	(b) Buckling with node at stiffener	

1	1	9.44	26.6	182
	10	11.65	28.4	144
	30	17.59	32.8	87
	100	33.55	52.0	55
	1000	157.40	228.0	45
1.5	1	7.12	13.9	95
	10	8.55	15.7	84
	30	14.30	21.7	52
	100	27.15	40.7	50
	1000	129.70	176.0	36
2	1	6.62	9.4	43
	10	7.65	11.7	52
	30	12.48	17.6	21
	100	26.19	33.6	28
	1000	117.30	157.4	34

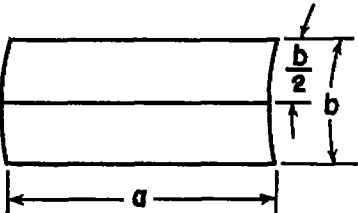
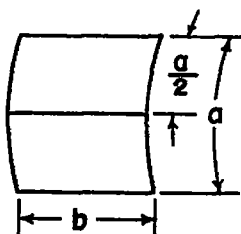
1.5	1	7.37	24.0	226
	10	9.49	25.6	170
	30	15.23	30.4	100
	100	30.73	50.0	63
	1000	154.00	216.0	40
2	1	6.61	23.2	251
	10	8.95	24.2	170
	30	14.10	28.5	103
	100	29.10	48.8	68
	1000	153.00	216.0	41

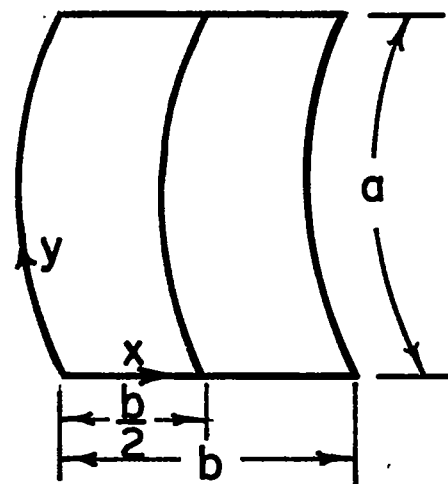
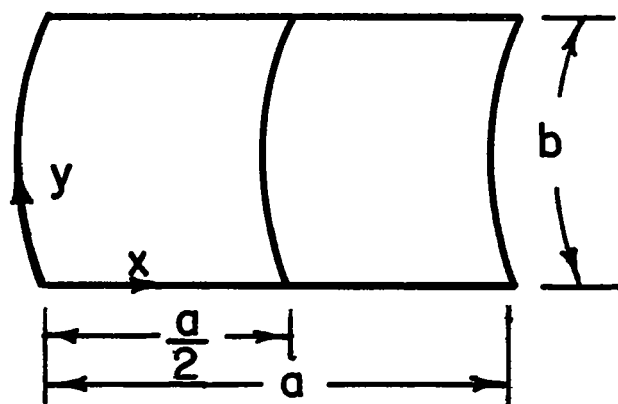
TABLE 1 - Concluded

CRITICAL-SHEAR-STRESS COEFFICIENTS FOR BUCKLING WITHOUT STIFFENER

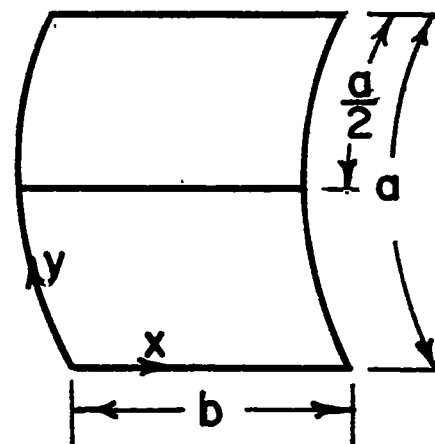
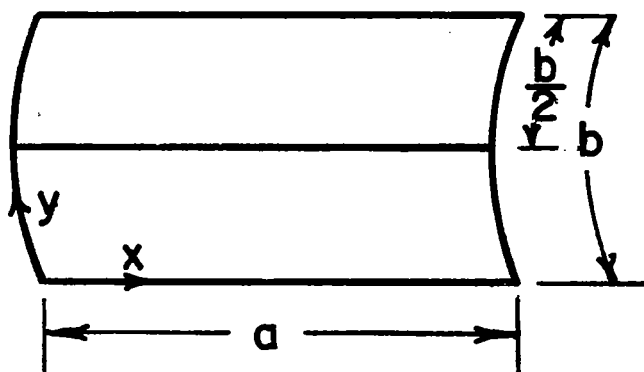
AND FOR BUCKLING WITH NODE AT STIFFENER OFFERING

ZERO TORSIONAL RESTRAINT - Concluded

$\frac{a}{b}$	$z = \frac{b^2}{rt} \sqrt{1 - \nu^2}$	$k_B = \frac{rtb^2}{D\pi^2}$		Percentage increase $\frac{(b) - (a)}{(a)} \times 100$
		(a) Buckling without stiffener	(b) Buckling with node at stiffener	
				
1	1	9.44	26.4	180
	10	11.65	26.6	128
	30	17.59	29.2	66
	100	33.55	45.2	35
	1000	157.40	182.5	16
1.5	1	7.12	23.7	234
	10	8.55	24.4	185
	30	14.30	27.2	90
	100	27.15	42.4	55
	1000	129.70	157.2	37
2	1	6.62	22.7	244
	10	7.65	23.2	203
	30	12.48	26.4	111
	100	26.19	40.8	56
	1000	117.30	149.2	27
				
1.5	1	7.37	13.6	85
	10	9.49	15.5	63
	30	15.23	20.3	33
	100	30.73	37.3	21
	1000	154.00	160.0	4
2	1	6.61	9.4	43
	10	8.95	11.7	30
	30	14.10	17.6	25
	100	29.10	33.6	15
	1000	153.00	157.4	3



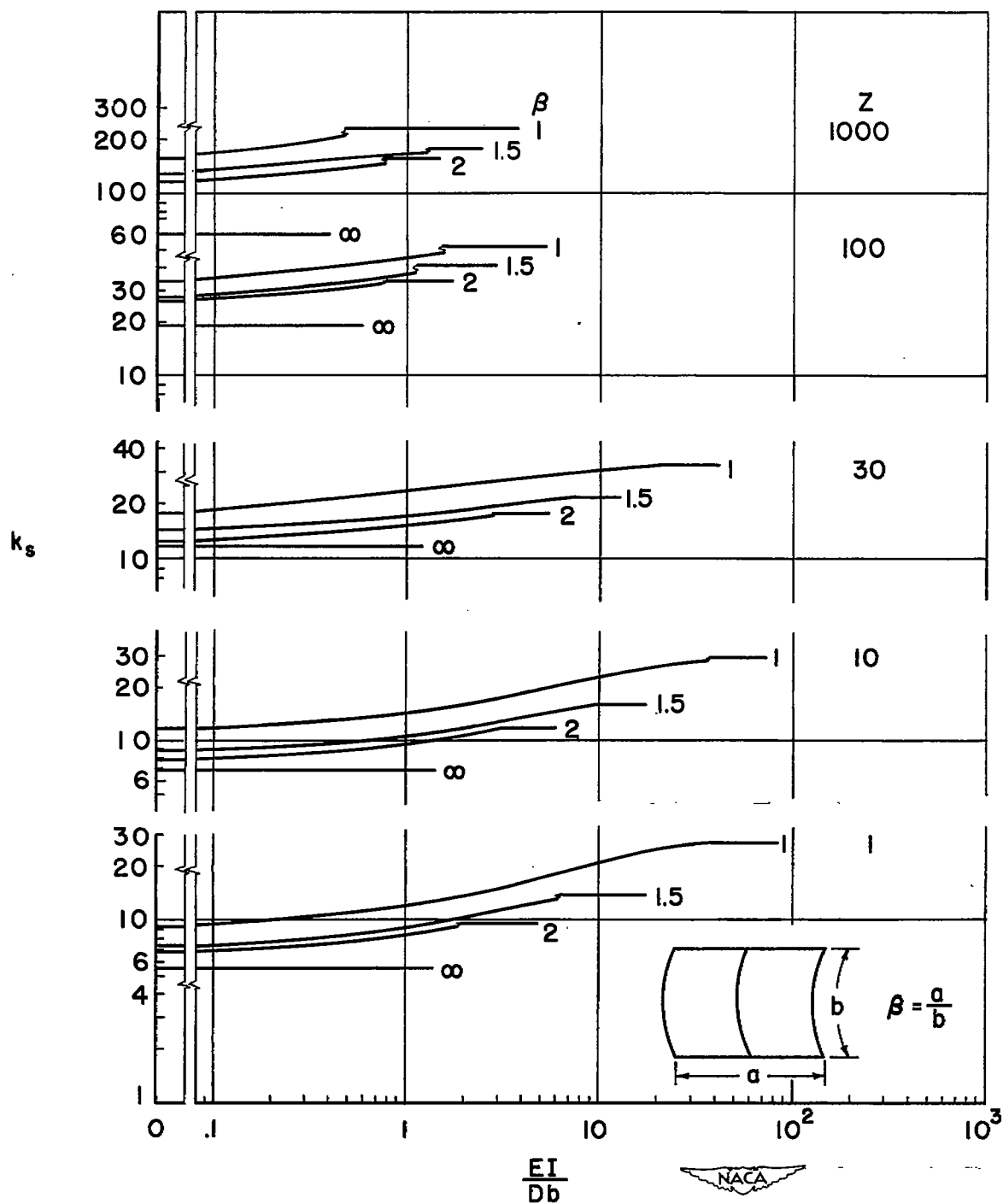
(a) Curved rectangular panel with central circumferential stiffener.



(b) Curved rectangular panel with central axial stiffener.

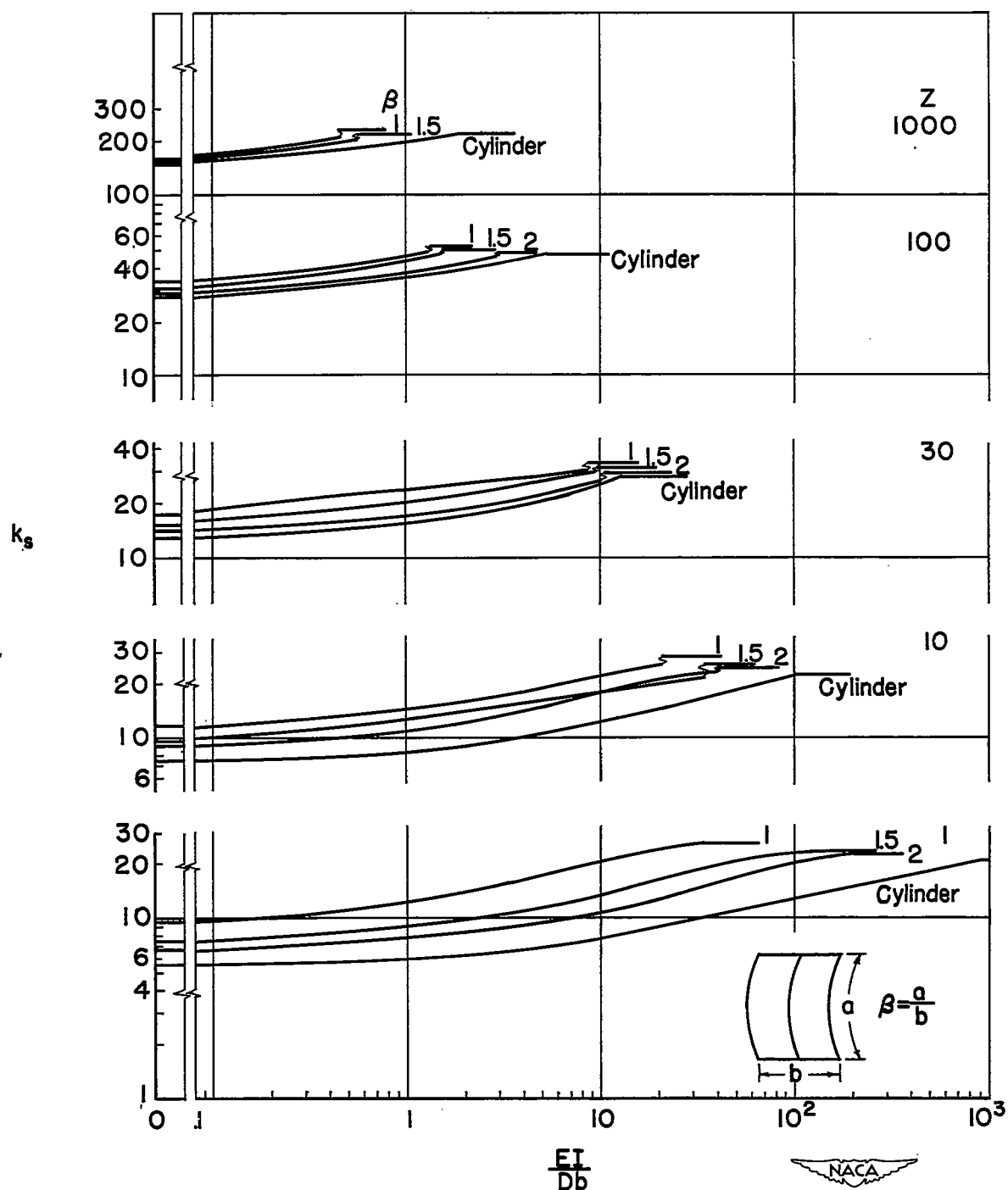


Figure 1.— Coordinate system used.



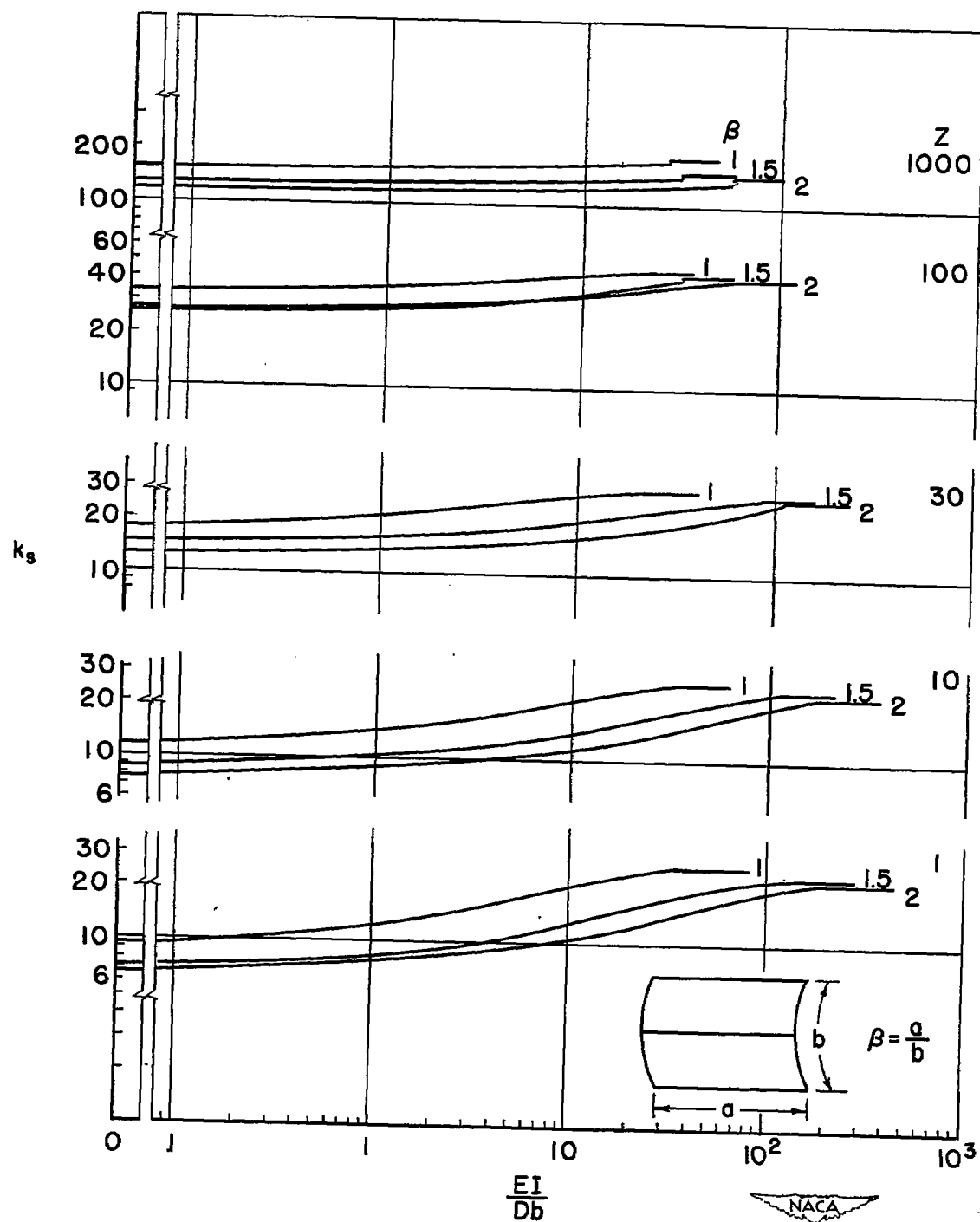
(a) Circumferential stiffener; axial dimension greater than circumferential dimension.

Figure 2.— Theoretical critical shear stress of a curved rectangular panel with a central stiffener.



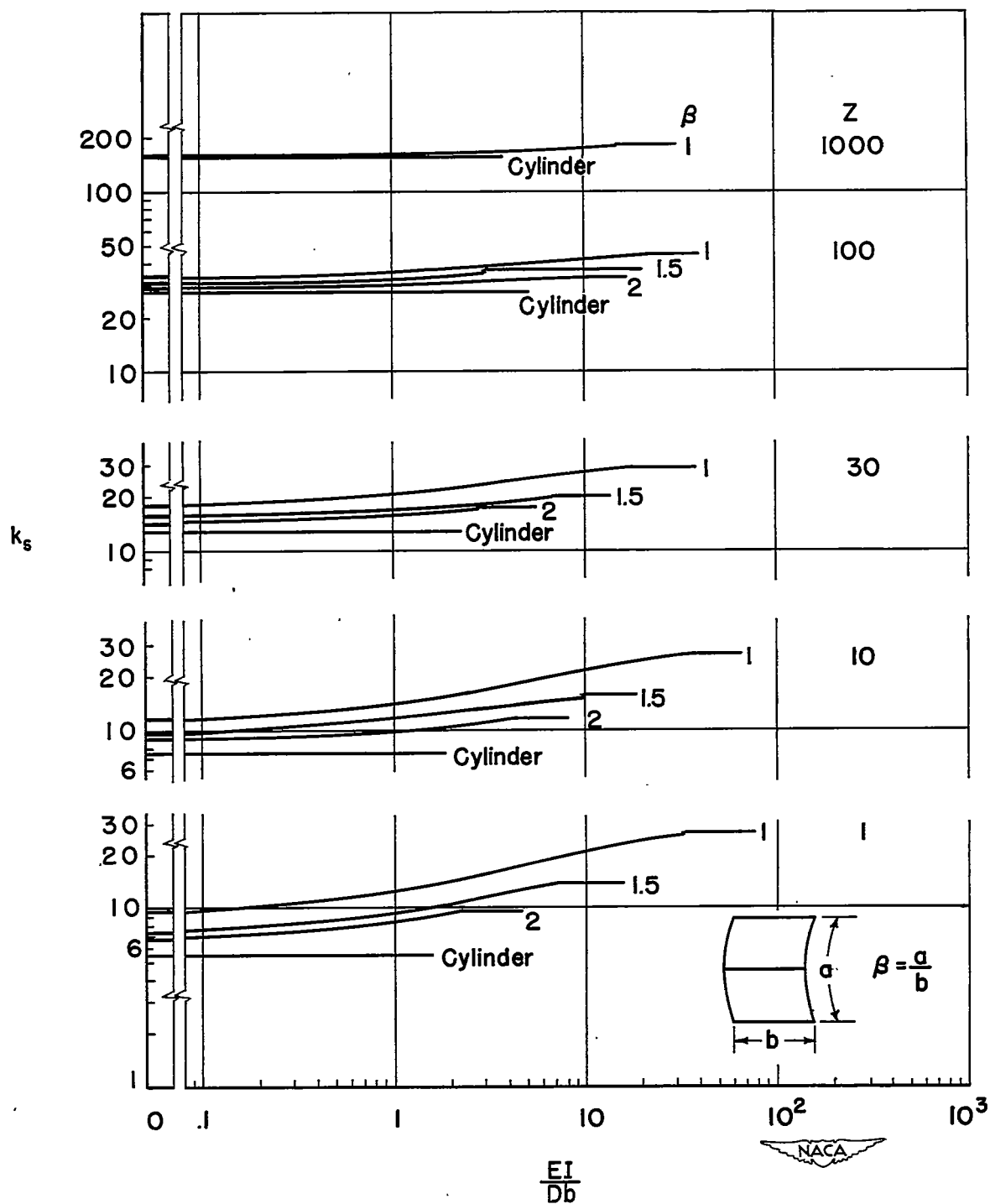
(b) Circumferential stiffener; circumferential dimension greater than axial dimension.

Figure 2.— Continued.



(c) Axial stiffener; axial dimension greater than circumferential dimension.

Figure 2.- Continued.



(d) Axial stiffener; circumferential dimension greater than axial dimension.

Figure 2.- Concluded.